Solution 10

1. Determine which of the following sets are dense, open dense, nowhere dense, of first category and residual in $\mathbb{R}$ (you may draw a table):

   (a) $A = \{n/2^m : \ n, m \in \mathbb{Z}\}$,
   (b) B, all irrational numbers,
   (c) $C = \{0, 1, 1/2, 1/3, \cdots\}$,
   (d) $D = \{1, 1/2, 1/3, \cdots\}$,
   (e) $E = \{x : \ x^2 + 3x - 6 = 0\}$,
   (f) $F = \cup_k (k, k+1), k \in \mathbb{N}$,

   Solution. (a) A is dense, not open, not nowhere dense, of first category and not residual.

   (b) B is dense, not open, not nowhere dense, of second category and residual.

   (c) C is not dense, not open (closed in fact), nowhere dense, of first category and not residual.

   (d) D is not dense, not open (not closed), nowhere dense, of first category and not residual.

   (e) E is the finite set $\{(-3+\sqrt{33})/2, (-3-\sqrt{33})/2\}$. It is not dense, not open (closed in fact), nowhere dense, of first category and not residual.

   (e) F is dense, open, not nowhere dense, of second category and residual.

Sets	Dense	Open dense	Nowhere dense	First category	Residual
A	✓	✗	✗	✓	✗
B	✓	✗	✗	✗	✓
C	✗	✗	✓	✓	✗
D	✗	✗	✓	✓	✗
E	✗	✗	✓	✓	✗
F	✓	✓	✗	✗	✓

2. Determine which of the following sets are dense, open dense, nowhere dense, of first category and residual in $C[0, 1]$ (you may draw a table):

   (a) $\mathcal{A}$, all polynomials whose coefficients are rational numbers,
   (b) $\mathcal{B}$, all polynomials,
   (c) $\mathcal{C} = \{f : \ \int_0^1 f(x)dx \neq 0\}$,
   (d) $\mathcal{D} = \{f : \ f(1/2) = 1\}$.

   Solution. (a) $\mathcal{A}$ is dense (and countable too), not open, not nowhere dense, of first category, and not residual.

   (b) $\mathcal{B}$ is dense (and uncountable), not open, not nowhere dense, of first category and not residual. ($\mathcal{B}$ can be expressed as the countable union of P_n where P_n is the set of all polynomials of degree not exceeding n. Each P_n is closed and nonwhere dense.)

   (c) $\mathcal{C}$ is dense, open, not nowhere dense, of second category, and residual.

   (d) $\mathcal{D}$ is not dense, not open (closed in fact), nowhere dense, of first category, and not residual.

Sets	Dense	Open dense	Nowhere dense	First category	Residual
$\mathcal{A}$	✓	✗	✗	✓	✗
$\mathcal{B}$	✓	✗	✗	✓	✗
$\mathcal{C}$	✓	✓	✗	✗	✓
$\mathcal{D}$	✗	✗	✓	✓	✗

3. Use Baire category theorem to show that transcendental numbers are dense in the set of real numbers.

Solution. A number is called algebraic if it is a root of some polynomial with integer coefficients and it is transcendental otherwise. Let $\mathcal{A}$ be all algebraic numbers and $\mathcal{T}$ be all transcendental numbers. We know that $\mathcal{A}$ is a countable set $\{a_j\}$. Let $\mathcal{A}_n = \{a_1, \cdots, a_n\}$ so $\mathcal{A} = \bigcup_n \mathcal{A}_n$ is a countable union of closed and nowhere dense sets $\mathcal{A}_n$. Hence $\mathcal{A}$ is of first category. As $\mathcal{T}$ is the complement of $\mathcal{A}$, it is a residual set. Since $\mathcal{R}$ is complete, $\mathcal{T}$ is dense by Baire category theorem.

Alternatively, you may argue that the complement of each $\mathcal{A}_n$ is open and dense, and since $\mathcal{T}$ is the intersection of all these complements, by Baire category theorem, any countable intersection of open dense sets in a complete metric space is dense, hence $\mathcal{T}$ is dense.

4. A point p in a metric space X is called an *isolated point* if there is an open set G such that $G \cap X = \{p\}$, that is, $\{p\}$ is open. A set E in X is a *perfect set* if it is closed and contains no isolated points.

   (a) For each x in the perfect set E, there exists a sequence in E consisting of infinitely many distinct points converging to x.

   (b) Every perfect set is uncountable in a complete metric space.

Solution. (a). For each $n \geq 1$, as $(B_{1/n}(x) \setminus \{x\}) \bigcap E$ is nonempty, we pick a point from it to form $\{x_n\}$. Obviously, there are infinitely many distinct points in this sequence and it converges to x as $n \to \infty$.

(b). Assume on the contrary that the perfect set E is countable, $E = \{a_n\}, n \geq 1$. We have $E = \bigcup_{n=1}^{\infty} \{a_n\}$. Obviously every $\{a_n\}$ is a closed set. On the other hand, every ball containing a_n must contain some points different from a_n. We conclude that every $\{a_n\}$ is a closed set with empty interior. However, by assumption, (E, d) is a complete metric space. By Baire Category Theorem E cannot have such decomposition. Therefore, it must be uncountable.

Note. Applying to $\mathbb{R}$, it gives another proof that $\mathbb{R}$ is uncountable.

5. Let f be a real-valued function on $\mathbb{R}$. Define the oscillation of f at x to be $\omega_f(x) = \lim_{\delta \to 0^+} \omega_f(x, \delta)$ where

$$\omega_f(x, \delta) = \sup\{|f(y) - f(z)| : y, z \in (x - \delta, x + \delta)\} .$$

   (a) For each $\rho > 0$, the set $D = \{x : \ \omega_f(x) \geq \rho\}$ is closed.

   (b) Show that the set of all discontinuous points of f is given by $\bigcup_n D_n$ where $D_n = \{x : \omega_f(x) \geq 1/n\}$.

   (c) Show that we cannot find a function which is discontinuous exactly at all irrational numbers.

Solution (a) As $(x-\delta_1, x+\delta) \subset (x-\delta_2, x+\delta)$ for $\delta_1 < \delta_2$, $\omega_f(x,\delta)$ is decreasing in δ. Hence $\omega_f(x,\delta) \geq \rho$ for all $x \in D$. Let $x_n \in D, x_n \to x$. We want to show $x \in D$. Given $\delta > 0$, for large n, x_n is contained in $(x-\delta, x+\delta)$, hence there exists a small δ' such that $(x_n - \delta', x_n + \delta') \subset (x-\delta, x+\delta)$. It follows that $\omega_f(x,\delta) \geq \omega_f(x_n, \delta') \geq \rho$. Since this is valid for all δ, $\omega_f(x) \geq \rho$, that is, $x \in D$, so D is closed.

(b) Obvious.

(c) Suppose there is a function f which is exactly discontinuous on $\mathbb{I}$. By (b), $\mathbb{I} = \bigcup_n D_n$. Since $\mathbb{I}$ does not contain any ball, each D_n cannot contain any ball. By (a) we conclude that $D_n = \overline{D_n}$ is nowhere dense. On the other hand, write $\mathbb{Q} = \bigcup_n \{q_n\}$ where each $\{q_n\}$ is closed and nowhere dense and express

$$\mathbb{R} = \mathbb{I} \cup \mathbb{Q} = \bigcup D_n \cup \bigcup \{q_n\}$$

which shows that $\mathbb{R}$ is a countable union of nowhere dense sets, contradicting its completeness (Corollary 4.11).

Note Recall in MATH2060 we learned that the Thomae function is continuous at all irrational numbers but discontinuous at all rational numbers. It is natural to wonder if there is function doing the opposite things. This exercise gives a negative answer.

6. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$.
   (a) Show that $\|x\| \leq C\|x\|_2$ for some C where $\|\cdot\|_2$ is the Euclidean metric.
   (b) Deduce from (a) that the function $x \mapsto \|x\|$ is continuous with respect to the Euclidean metric.
   (c) Show that the inequality $\|x\|_2 \leq C'\|x\|$ for some C' also holds. Hint: Observe that $x \mapsto \|x\|$ is positive on the unit sphere $\{x \in \mathbb{R}^n : \ \|x\|_2 = 1\}$ which is compact.
   (d) Establish the theorem asserting any two norms in a finite dimensional vector space are equivalent.

Solution. In the following we prove more generally any two norms $\|\cdot\|$ and $\|\cdot\|'$ on a finite dimensional vector space V are equivalent, that is, there exist $\alpha, \beta > 0$ such that $\alpha\|v\| \leq \|v\|' \leq \beta\|v\|$ for all v. Fix a basis $\{v_1, \cdots, v_n\}$ so that each v can be uniquely represented as $\sum_{j=1}^n a_j v_j$. It is easy to verify $\|v\|_2 = \sqrt{\sum_j a_j^2}$ defines a norm on V. It suffices to compare all other norms with this "Euclidean one". We claim there exist $c_1, c_2 > 0$ such that

$$c_1\|v\|_2 \leq \|v\| \leq c_2\|v\|_2, \quad \forall v \ .$$

First, by Cauchy-Schwarz inequality,

$$\|v\| = \|\sum_j a_j v_j\| \leq \sum_j |a_j|\|v_j\| \leq \sqrt{\sum_j a_j^2}\sqrt{\sum_j \|v_j\|^2} \equiv c_2\|v\|_2 \ .$$

Next, let $\varphi(v) = \|v\|$. We have

$$|\varphi(v) - \varphi(w)| = |\|v\| - \|w\|| \leq \|v - w\| \leq c_2\|v - w\|_2 \ .$$

Therefore, as $v_n \to v$ in $\|\cdot\|_2, \varphi(v_n) \to \varphi(v)$, which shows that φ is continuous in the $\|\cdot\|_2$-norm. As the unit sphere $S = \{v \in V : \ \|v\|_2 = 1\}$ is closed and bounded, φ attains its minimum over S at some v_0, so $\varphi(v) = \|v\| \geq \varphi(v_0) > 0, \ \forall v \in S$. For any non-zero vector v, $v/\|v\|_2 \in S$. Therefore, $\|v/\|v\|_2\| \geq \varphi(v_0)$, that is, $\|v\| \geq c_1\|v\|_2$ where $c_1 = \varphi(v_0)$.

Note This problem is used in the proof of Theorem 4.15.

7. Let P be the vector space consisting of all polynomials. Show that we cannot find a norm on P so that it becomes a Banach space.

   Solution. Let P_n be the vector subspace of P consisting of all polynomials of degree less than or equal to n. Then $P = \bigcup_{n=1}^{\infty} P_n$. Any norm on P_n is equivalent to the "Euclidean norm": $\|p\|_2 = (\sum_{k=0}^{n} a_k^2)^{1/2}$ when $p(x) = \sum_{k=0}^{n} a_k x^k$. Using this fact, one can show that P_n is a closed subspace of P in any norm. On the other hand, it is clear that P_n is nowhere dense. By Baire category theorem, it is impossible to decompose P as a union of nowhere dense sets when its induced metric is complete.

8. Let $\mathcal{F}$ be a subset of $C(X)$ where X is a complete metric space. Suppose that for each $x \in X$, there exists a constant M depending on x such that $|f(x)| \leq M,\ \forall f \in \mathcal{F}$. Prove that there exists an open set G in X and a constant C such that $\sup_{x\in G} |f(x)| \leq C$ for all $f \in \mathcal{F}$. Suggestion: Consider the decomposition of X into the sets $X_n = \{x \in X : |f(x)| \leq n,\ \ \forall f \in \mathcal{F}\}$.

   Solution. By assumption, $X = \bigcup_n X_n$. It is clear that each X_n is closed. By the completeness of X we appeal to Baire Category Theorem to conclude that there is some n_1 such that X_{n_1} has non-empty interior, call it G. Then $|f(x)| \leq n_1,\ \forall x \in G$, for all $f \in \mathcal{F}$.